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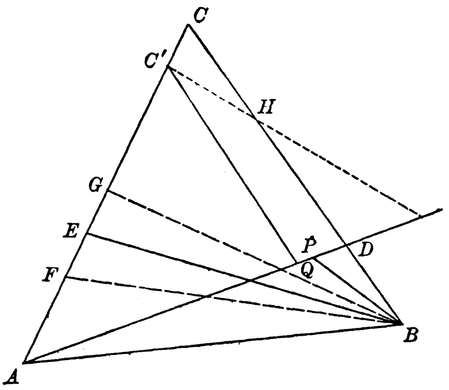
PROOF OF A THEOREM CONCERNING ISOSCELES TRIANGLES[*]

BY H. F. BLICHFELDT

THE theorem " a triangle is isosceles if the bisectors of two of its angles are equal," has attracted considerable attention, owing to the seeming difficulty of its proof. The writer has discovered a proof that covers the case of the internal trisectors, etc., as well as the bisectors, and holds as well for the so-called non-euclidean as for the euclidean space; *i. e.* it does not depend upon parallel lines, nor on the angle-sum of a triangle being any given amount. Although numerous proofs have been given in various books and periodicals,[†] the writer submits herewith the proof referred to, as it may possess some interest on account of its somewhat more general character.



The theorem may be stated thus: *In a triangle ABC, straight lines AD and BE are drawn, intersecting the sides BC and AC respectively in the points D and E, and dividing the angles CAB and CBA internally in the same ratio, so that*

angle CAD : angle DAB = angle CBE : angle EBA;

then, if BE and AD are equal, the triangle ABC is isosceles.

This theorem is true whether the triangle be drawn in euclidean or in noneuclidean space, provided it can be drawn entirely within a certain limited region of the space considered.

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[*] This article is a modification of part of a paper: Demonstration of a Pair of Theorems in Geometry, by the same author, read before the Edinburgh Math. Soc., Dec. 13, 1901.

[†] *e. g.* in G. B. Halsted's *Elementary Synthetic Geometry*, pp. 44–45. Some references are given in J. S. Mackay's *The Elements of Euclid*, Edin., 1888, p. 109.

The proof will be based on the following propositions only, — aside from the pan-geometric axioms as to superposition, etc.*

Within a certain limited region of space,

1. Two triangles are congruent if a side and two adjacent angles of one are respectively equal to a side and two adjacent angles of the other.
2. The exterior angle of a triangle is greater than either opposite interior angle.
3. If two sides of a triangle are unequal, opposite the greater side lies the greater angle.
4. One, and only one, perpendicular can be drawn from any given point to a given straight line.
5. If oblique lines drawn from a point in a perpendicular to a straight line cut off unequal distances from the foot of the perpendicular, the more remote is the greater.

To prove our theorem it is sufficient to prove that if the triangle ABC is not isosceles, the lines BE and AD are unequal. This we shall proceed to do.

Given, in triangle ABC,

$$AC > BC,$$
$$\text{angle } CAD : \text{angle } DAB = \text{angle } CBE : \text{angle } EBA;$$

to prove $AD > BE$.

Since $AC > BC$, angle $B >$ angle A (Prop. 3). Hence:

$$\text{angle } CBE > \text{angle } CAD, \qquad \text{angle } EBA > \text{angle } DAB.$$

Draw BG and BF making the angles CBG and FBA respectively equal to the angles CAD and DAB, and meeting AC in the points G and F respectively.

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* See H. P. Manning, *Non Euclidean Geometry*, for the proof of these propositions, and for the immediate assumptions underlying them. In chapter 1 are given all the propositions referred to. It may be noticed that the proof holds if the triangle be drawn on a certain limited portion of any surface of constant curvature — the sphere for instance — in ordinary euclidean space. By "straight line" we must then understand "geodesic," and the word "congruent" must be replaced by the phrase "congruent or symmetric" in proposition 1, and by the word "symmetric" wherever it occurs in the subsequent demonstration. The term "symmetric" is used here in the sense in which it is generally used in spherical geometry.

a. To prove $BF < AD$.

Locate a point P in AD so that angle PBA = angle A. P lies between A and D, as angle $B >$ angle A. Hence $AP < AD$. Now, the triangles APB and BFA are congruent (Prop. 1). Hence, $BF = AP < AD$.

β. Again, to prove $BG < AD$.

Locate C' on AC so that $AC' = BC$. The point C' lies between A and C, as $AC > BC$. Draw $C'Q$ making angle $AC'Q$ = angle ACD. The lines $C'Q$ and AD will meet at Q, say, forming a triangle $AC'Q$ congruent with the triangle BCG (Prop. 1).

The lines $C'Q$ and CD, extended if need be, cannot intersect in the limited region of the plane considered. If they met at H, say, a point within the region considered, we should have in this region a triangle $HC'C$ having an external angle, $AC'H$ equal to an opposite interior angle, ACH, which is impossible by Prop. 2.

Thus Q must lie between A and D, so that $AQ < AD$. But $AQ = BG$, as the triangles $AC'Q$ and BCG are congruent. Hence, $BG = AQ < AD$.

γ. Finally, to prove BE either less than BF or less than BG.

Drop the perpendicular from B upon the line AC (Prop. 4). Then, one at least of the oblique lines BG and BF will cut off a greater distance from the foot of the perpendicular on the line AC than will BE. Hence, by Prop. 5, BE is either less than BF or less than BG.

By (a) and (β), each of the latter is less than AD. Hence, $AD > BE$.

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